ON ECONOMIC MODELS SOLVABLE BY THE EXPONENTIAL CASE OF THE LEAST SQUARES METHOD

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ABSTRACT: *The purpose of this paper is to present and exemplify how the exponential case of the least square method can have technical-economic applications. A general presentation of the method is presented beforehand with classic examples of its use. A potential economic example is presented later together with the concrete application of the method to solve and obtain the result.*

KEY WORDS: *exponential model, least squares method, continuous capitalization.*

JEL CLASSIFICATION: *J61.*

1. INTRODUCTION

The present work represents a continuation of the results obtained and presented by the author in Mitran (2003) and which are related to various applications of different cases of the so-called *least squares method.*

Suppose that in a certain context is obtained a set of data $\{(x_i, y_i), i = 1, \dots, n\}$ for which an approximate function must be found under certain conditions.

Definition 1 *We call least squares methods those methods by which, under certain given conditions, a function y is determined provided that*

$$\sum_{i=1}^{n} (y(x_i) - y_i)^2 \rightarrow minim$$

Probably the best-known linear approximation model using the least squares method is the one is the linear one in which

$$y(x) = ax + b$$

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but there are also numerous cases where the function *y* is of parabolic type

$$y(x) = ax^2 + bx + c$$

such as, for example, the approach described by Mitran (2003), some of whose results are also used in the present paper.

2. Theoretical aspects

In the exponential model in the least squares method the function *y* has the form:

$$y(x) = \alpha \cdot e^{\beta x}, \alpha, \beta > 0, \beta \neq 1 \qquad (1)$$

or

$$y(x) = \alpha \cdot \beta^{x}, \alpha, \beta > 0, \beta \neq 1 \qquad (2)$$

As a first eloquent example we have *the formula of radioactive decay* which says that:

$$N(t) = N_0 \cdot e^{-\lambda t}$$

where

- N_0 is the initial number of nuclei;

- λ represents the decay constant.

This exponential law shows the decay over time of a radioactive source.

A second such example comes from the economic field. From an economic point of view, we have *the problem of continuous capitalization*. In the case of continuous capitalization, we have the calculation formula:

$$S(t) = S_0 \cdot e^{rt}$$

where:

-*t* represents the investment period;

-S_0 represents the initial amount;

-*r* represent the interest rate;

-$S(t)$ represents the value of the investment after time *t;*

As can be clearly seen from the previous relation, the value of the investment after time *t* depends on the initial amount and from the interest rate. What we propose next, which is also the purpose of this study, is to find a solution for determining the value of the investment after time *t* without having to know the initial amount and the interest rate.

For this we will use the least squares method, the exponential case, in which we will use the model given by relation (2). Until we get to this, however, we will present in parallel the solution methods for the least squares approximation models described by both relation (1) and relation (2).

In both case (1) and case (2) we proceed by logarithm, applying the natural logarithm in both cases.

In case (1) we obtain:

$$\ln y(x) = \ln \alpha + \beta \cdot x \tag{3}$$

and then multiplying equation (3) by x we find

$$x \cdot \ln y\,(x) = x \cdot \ln \alpha + \beta \cdot x^2 \tag{4}$$

This means that

$$\ln x_i = \ln \alpha + \beta \cdot x_i\,, \forall i = \overline{1,n} \tag{5}$$

respectively that

$$\mathrm{x_i} \cdot \ln y(x_i) = x_i \cdot \ln \alpha + \beta \cdot {x_i}^2, \forall i = \overline{1,n} \tag{6}$$

In this way we will obtain the system

$$\begin{cases} \sum_{i=1}^{n} \ln y(x_i) = n \cdot \ln \alpha + \beta \cdot \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i \ln y(x_i) = \ln \alpha \cdot \sum_{i=1}^{n} x_i + \beta \cdot \sum_{i=1}^{n} {x_i}^2 \end{cases} \tag{7}$$

In the work described by Mitran (2003), a procedure for linearizing data was given using, among other things, a system of partial differential equations. Without going into too much detail, we will just say that, taking into account of the relations (7) and everything that we have just mentioned we will find the system:

$$\begin{cases} \sum_{i=1}^{n} \ln y_i = n \cdot \ln \alpha + \beta \cdot \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} \mathrm{x_i} \ln y_i = \ln \alpha \cdot \sum_{i=1}^{n} x_i + \beta \cdot \sum_{i=1}^{n} {x_i}^2 \end{cases} \tag{8}$$

Similarly, in case (2) we will obtain the relation

$$\ln y(x) = \ln \alpha + x \cdot \ln \beta \tag{9}$$

and from here, proceeding similarly by multiplying by x we find

$$x \cdot \ln y(x) = x \cdot \ln \alpha + x^2 \cdot \ln \beta \tag{10}$$

We deduce that

$$\ln y(x_i) = \ln \alpha + x_i \cdot \ln \beta\,, \forall i = \overline{1,n} \tag{11}$$

and

$$x_i \cdot \ln y(x_i) = x_i \cdot \ln \alpha + {x_i}^2 \cdot \ln \beta\,, \forall i = \overline{1,n} \tag{12}$$

Similarly, we will obtain a new system of the form

$$\begin{cases} \sum_{i=1}^{n} \ln y(x_i) = n \cdot \ln \alpha + \ln \beta \cdot \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i \ln y(x_i) = \ln \alpha \cdot \sum_{i=1}^{n} x_i + \ln \beta \cdot \sum_{i=1}^{n} {x_i}^2 \end{cases} \tag{13}$$

Similarly, taking into account of the relations (13) and what I previously presented regarding the results obtained in Mitran (2003), we will find similarly to the result obtained in (8) the system:

$$\begin{cases} \sum_{i=1}^{n} \ln y_i = n \cdot \ln \alpha + \ln \beta \cdot \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i \ln y_i = \ln \alpha \cdot \sum_{i=1}^{n} x_i + \ln \beta \cdot \sum_{i=1}^{n} {x_i}^2 \end{cases} \tag{14}$$

In the case of the system of equations given by relation (8) the unknows are $\ln \alpha$ and β and its determinant has the value:

$$\Delta = \begin{vmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} {x_i}^2 \end{vmatrix} > 0 (\neq 0) \tag{15}$$

and the solutions of the system will be given by

$$\ln \alpha = \frac{\Delta_1}{\Delta}, \beta = \frac{\Delta_2}{\Delta} \tag{16}$$

where

$$\Delta_1 = \begin{vmatrix} \sum_{i=1}^{n} \ln y_i & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i \ln y_i & \sum_{i=1}^{n} {x_i}^2 \end{vmatrix} \tag{17}$$

and

$$\Delta_2 = \begin{vmatrix} n & \sum_{i=1}^{n} \ln y_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i \ln y_i \end{vmatrix} \tag{18}$$

In the case of the system of equations given by relation (13) the unknows are $\ln \alpha$ and $\ln \beta$ and its determinant has the same value as in the case given by (8):

$$\Delta = \begin{vmatrix} n & \displaystyle\sum_{i=1}^{n} x_i \\ \displaystyle\sum_{i=1}^{n} x_i & \displaystyle\sum_{i=1}^{n} {x_i}^2 \end{vmatrix} > 0 (\neq 0)$$

and the solutions of the system will be given by

$$\ln \alpha = \frac{\Delta_1}{\Delta}, \ln \beta = \frac{\Delta_2}{\Delta} \tag{19}$$

with the same values

$$\Delta_1 = \begin{vmatrix} \sum_{i=1}^{n} \ln(x_i) & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i \ln(x_i) & \sum_{i=1}^{n} {x_i}^2 \end{vmatrix}$$

and

$$\Delta_2 = \begin{vmatrix} n & \sum_{i=1}^{n} \ln y_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i \ln y_i \end{vmatrix}$$

given by (15) and (16).

To solve systems of type (8) or (13) it is advisable to create a table of the following type:

Table 1

i	x_i	y_i	${x_i}^2$	$\ln(y_i)$	$x_i \ln(y_i)$
1	x_1	y_1	${x_1}^2$	$\ln(y_1)$	$x_1 \ln(y_1)$
2	x_2	y_2	${x_2}^2$	$\ln(y_2)$	$x_2 \ln(y_2)$
…	…	…	…		
n	x_n	y_n	${x_n}^2$	$\ln(y_n)$	$x_n \ln(y_n)$
Σ	$\sum_{i=1}^{n} x_i$		$\sum_{i=1}^{n} {x_i}^2$	$\sum_{i=1}^{n} \ln(y_i)$	$\sum_{i=1}^{n} x_i \ln(y_i)$

Remark 1 Such a type of table was used by the author in the work given by Mitran (2003). Of course, such a table is symbolic, presenting only the values that need to be entered and the operations that need to be performed, and then the concrete values need to be entered and the required results calculated, all of this of course in another table.

Remark 2 Once the solutions of systems of type (8) and (13) have been found, which means finding solutions in which values such as $\ln \alpha$ or $\ln \beta$ will appear. Finding the values we specifically need can be done using the usual tables of mathematical formulas or, more simply, using the functions of a scientific calculator.

3. NUMERICAL EXAMPLE

Let us assume that a company can obtain certain information about the performance of a competing company about the amounts resulting in its profit in certain time intervals within a certain period of time. The problem arises of determining a calculation formula that approximates the evolution of the continuous capitalization of the amounts obtained by the competing firm.

The values in the following table for x_i are expressed in time interval and those for y_i in thousands of monetary units.

The time interval considered is not necessarily specified, nor is the type of unitary currency, these being irrelevant in the context of the present study.

Table 2

x_i	y_i
0,06	4,55
0,07	4,61
0,1	4,8
0,12	4,93
0,15	5,13
0,17	5,27
0,21	5,56
0,23	5,71

Without going into too much detail, we will just say that based on the calculation and research carried out we are led to an approximation of the form (2). In this case the symbolic values in table (1) are transformed, if we take into account the data in table (2), in:

Table 3

i	x_i	y_i	${x_i}^2$	$\ln(y_i)$	$x_i \ln(y_i)$
1	0,06	4,55	0,0036	1,5151	0,0909
2	0,07	4,61	0,0049	1,5282	0,1070
3	0,1	4,8	0,01	1,5686	0,1569
4	0,12	4,93	0,0144	1,5953	0,1914
5	0,15	5,13	0,0225	1,6351	0,2452
6	0,17	5,27	0,0289	1,6620	0,2825
7	0,21	5,56	0,0441	1,7156	0,3603

8	0,23	5,71	0,0529	1,7422	0,4007
∑	1,11		0,1813	12,9621	1,8349

In this case the system given by (13) becomes:

$$\begin{cases} 8 \cdot \ln\alpha + 1{,}11 \cdot \ln\beta = 12{,}9622 \\ 1{,}11 \cdot \ln\alpha + 0{,}1813 \cdot \ln\beta = 1{,}8349 \end{cases} \quad (19)$$

From relations (13), based on appropriate substitutions, we obtain

$$\Delta = \begin{vmatrix} 8 & 1{,}11 \\ 1{,}11 & 0{,}1813 \end{vmatrix} = 0{,}2183$$

and then from relations (15) and (16), proceeding similarly, we obtain the values

$$\Delta_1 = \begin{vmatrix} 12{,}9622 & 1{,}11 \\ 1{,}8349 & 0{,}1813 \end{vmatrix} \approx 0{,}3133$$

and

$$\Delta_2 = \begin{vmatrix} 8 & 12{,}9622 \\ 1{,}11 & 1{,}8349 \end{vmatrix} \approx 0{,}2912$$

and finally, taking care of (17), we will obtain for $\ln\alpha$ and $\ln\beta$ the values

$$\ln\alpha \approx 1{,}4352, \ln\beta \approx 1{,}3337 \quad (20)$$

From (19) we will find for α and β the approximate values

$$\alpha = e^{1{,}4352} \approx 4{,}1996, \beta = e^{1{,}3337} \approx 3{,}7961 \quad (21)$$

It can be seen that the values thus obtained 4,1996 for α and 3,7961 for β can be even better approximated by the simplified values 4,2 for α and 3,8 for β, in this way obtaining the final approximation corresponding to the relation

$$y(x) = 4{,}2 \cdot 3{,}8^x \quad (22)$$

Remark 3 Obtaining a formula such as the one given by relation (22) will subsequently allow us to anticipate any result we need related to the evolution of the amount obtained through continuous capitalization.

4. CONCLUSIONS

If in the classic examples of solving simple concrete economic problems of continuous capitalization the exponential model of type $\mathrm{S(t)} = \mathrm{S_0} \cdot \mathrm{e^{rt}}$ is used in which the value of the initial investment and the interest rate must be known, in the example proposed and solved in this paper a second case of exponential model is proposed and used for solving.

The advantage of this second type of exponential approximation is that it provides approximate results for continuous compounding without the need to know the initial capital and interest rate.

REFERENCES:

[1]. Despa, R., Vişan, C. (2003) *Mathematics applied in economics*, Sylvi Publishing House, Bucharest

[2]. Mitran, I. (2003) *Modeling of technical and economic decision*, AGIR Publishing House, Bucharest

[3]. Mitran, C.I. (2023) *On some economic models using the parabolic model of the least squares method*, Annals of the University of Petroşani, Economics, volume XXIII,

[4]. Mitran I., Mitran C.I. (2002) *Matematici aplicate în economie 1 și 2*, Editura Focus Petroșani